ME 261: Numerical Analysis

Lecture-11: Numerical Interpolation

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Lagrange Interpolating polynomial

- The Lagrange interpolating polynomial is simply a **reformulation** of the Newton polynomial that avoids the computation of divided differences.
- It can be represented concisely as:

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

- **where Π designates the "product of" the following terms**

Lagrange Interpolating polynomial

- **For Linear case (n=1) Lagrange interpolating polynomial:** .

$$f_1(x) = \sum_{i=0}^{1} L_i(x) f(x_i)$$

$$= L_0(x) f(x_0) + L_1(x) f(x_1)$$

$$L_0(x) = \frac{(x - x_1)}{(x_0 - x_1)}$$

$$L_1(x) = \frac{(x - x_0)}{(x_1 - x_0)}$$

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

Lagrange Interpolating polynomial

- **For quadratic case (n=2) Lagrange interpolating polynomial:**

$$f_2(x) = \sum_{i=0}^{2} L_i(x) f(x_i)$$

$$= L_0(x) f(x_0) + L_1(x) f(x_1) + L_2(x) f(x_2)$$

$$= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

Lagrange Interpolating polynomial

- **For cubic case (n=3) Lagrange interpolating polynomial:**

$$f_3(x)=\sum_{i=0}^{3}L_i(x)f(x_i)$$

$$=L_0(x)f(x_0)+L_1(x)f(x_1)+L_2(x)f(x_2)+L_3(x)f(x_3)$$

$$L_0(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}$$

$$L_1(x)=\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$L_2(x)=\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}$$

$$L_3(x)=\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

$$L_i(x)=\prod_{\substack{j=0\\ j\neq i}}^{n}\frac{x-x_j}{x_i-x_j}$$

Lagrange Interpolating polynomial

- **Fourth Order (n = 4) Lagrange interpolating polynomial:**

$$f_4(x) = \sum_{i=0}^{4} L_i(x) f(x_i)$$

$$= L_0(x) f(x_0) + L_1(x) f(x_1) + L_2(x) f(x_2) + L_3(x) f(x_3) + L_4(x) f(x_4)$$

$$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)}$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)}$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)}$$

$$L_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)}$$

$$L_4(x) = \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)}$$

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

Example

Construct a 4th order polynomial in Lagrange form that passes through the following points:

i	0	1	2	3	4
x_i	0	1	-1	2	-2
$f(x_i)$	-5	-3	-15	39	-9

Lagrange 4th order polynomial function:

$$f_4(x) = \sum_{i=0}^{4} L_i(x) f(x_i)$$

$$= L_0(x) f(x_0) + L_1(x) f(x_1) + L_2(x) f(x_2) + L_3(x) f(x_3) + L_4(x) f(x_4)$$

$$f_4(x) = -5L_0(x) - 3L_1(x) - 15L_2(x) + 39L_3(x) - 9L_4(x)$$

$$L_0(x)=\frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)}$$

$$L_1(x)=\frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)}$$

$$L_2(x)=\frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)}$$

$$L_3(x)=\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)}$$

$$L_4(x)=\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)}$$

i	0	1	2	3	4
x_i	0	1	-1	2	-2
$f(x_i)$	-5	-3	-15	39	-9

$$L_0(x) = \frac{(x-1)(x+1)(x-2)(x+2)}{(0-1)(0+1)(0-2)(0+2)} = \frac{(x-1)(x+1)(x-2)(x+2)}{4}$$

$$L_1(x) = \frac{x(x+1)(x-2)(x+2)}{(1-0)(1+1)(1-2)(1+2)} = \frac{x(x+1)(x-2)(x+2)}{-6}$$

$$L_2(x) = \frac{x(x-1)(x-2)(x+2)}{(-1-0)(-1-1)(-1-2)(-1+2)} = \frac{x(x-1)(x-2)(x+2)}{-6}$$

$$L_3(x) = \frac{x(x-1)(x+1)(x+2)}{(2-0)(2-1)(2+1)(2+2)} = \frac{x(x-1)(x+1)(x+2)}{24}$$

$$L_4(x) = \frac{x(x-1)(x+1)(x-2)}{(-2-0)(-2-1)(-2+1)(-2-2)} = \frac{x(x-1)(x+1)(x-2)}{24}$$

Derivation of Lagrange Form from Newton's Interpolating polynomial

- **Linear Interpolation**

$$\frac{f_1(x)-f(x_0)}{x-x_0}=\frac{f(x_1)-f(x_0)}{x_1-x_0}$$

$$f_1(x)=f(x_0)+\frac{f(x_1)-f(x_0)}{x_1-x_0}(x-x_0)$$

$$=f(x_0)-f(x_0)\frac{(x-x_0)}{(x_1-x_0)}+f(x_1)\frac{(x-x_0)}{(x_1-x_0)}$$

$$=f(x_0)\left(1-\frac{(x-x_0)}{(x_1-x_0)}\right)+f(x_1)\frac{(x-x_0)}{(x_1-x_0)}$$

$$=f(x_0)\frac{(-x+x_1)}{(x_1-x_0)}+f(x_1)\frac{(x-x_0)}{(x_1-x_0)}$$

$$=f(x_0)\frac{(x-x_1)}{(x_0-x_1)}+f(x_1)\frac{(x-x_0)}{(x_1-x_0)}$$

$$=L_0(x)f(x_0)+L_1(x)f(x_1)$$